On the Kostka-Green-Foulkes polynomials and Clebsch-Gordan numbers

A.N. KIRILLOV
Steklov Mathematical Institute, Leningrad
Fontanka 27, 191011 Leningrad, USSR

Dedicated to I.M. Gelfand
on his 75th birthday

Abstract. *An explicit formula for the Kostka-Green-Foulkes polynomials corresponding to bitableaux is presented.*

INTRODUCTION

It is well known that the eigenvectors of the Hamiltonian of a quantum integrable system may be constructed by using the so called Bethe Ansatz (see, for instance, [1]). Such vectors are called the Bethe vectors. They are parametrized by the solutions of a system of algebraic equations (the so called Bethe equations).

The creation of the quantum inverse scattering method made it clear that different integrable models are associated with different representations of the commutation relations between the matrix coefficients of the quantum monodromy matrix [2]. This idea was given a precise mathematical formulation in [3], [4]. It was shown that these matrix coefficients generate a Hopf algebra. Particular representations of this algebra give rise to integrable models. The states' space of the model coincides with the representation space.

Let g be a simple Lie algebra. An important class of integrable models are the so called g-invariant models. The states' space for such models is the tensor product of

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g-moduli : $\mathcal{M} = V_{\lambda_1} \otimes V_{\lambda_2} \otimes \ldots$ The Bethe vectors are the highest weight vectors in the irreducible components of $\mathcal{M}$ and are parametrized by the solutions of Bethe's equations.

It is a nontrivial question whether the Bethe vectors form a complete system. In the g-invariant case completeness means that the number of solutions of Bethe's equations must coincide with the number of g-irreducible components in $\mathcal{M}$. For $g = g\ell(n)$ and $\mathcal{M} = V_{\lambda_1} \otimes V_{\lambda_2} \otimes \ldots$ this has been proved analytically in [7] for the case when the representations V_{λ_i} correspond to rectangular Young diagrams λ_i. These latter representations are distinguished among all the other by the fact that in this case Bethe's vectors form an orthogonal system in $\mathcal{M}$.

Solutions of Bethe's equations are parametrized by special combinatorial objects, the so called rigged configurations. They were introduced in [5], [6] for the $su(2)$-case and in [7], [8] for the general one. The completeness of Bethe's vectors implies that g-irreducible components in $\mathcal{M}$ are parametrized by the Young tableaux (or bitableaux [11]). Hence in this case there should exist a combinatorial correspondence between the rigged configurations and the Young tableaux (bitableaux). Physically, such bijection naturally arises as one of the lattice's sites goes to infinity; this causes a restructuring of physical states, and the bijection should reflect these changes. In other words, the correspondence we are looking for is defined by the ramification rules for Bethe's vectors as one of the lattice's sites goes to infinity.

A bijection between the set of the standard Young tableaux (bitableaux) and the set of rigged configurations based on the study of the asymptotic behaviour of Bethe's vectors is established in [9], [10]. The present paper deals with some combinatorial aspects of this correspondence. Unfortunately, we were obliged to discard completely the physical background of the results obtained.

A few words are in order on the contents of the paper. In Section1 we recall the basic definitions and the necessary facts concerning the combinatorics of the Young tableaux. In particular, we review the definitions of the standard tableaux and bitableaux. In Section 2 we describe the decomposition into ireducible components of the tensor product of the representations of $g\ell(n)$ corresponding to rectangular Young diagrams. As a corollary we present a formula for the multiplicity of a weight for irreducible tensor represetations of the Lie supergroup $GL(N|M)$. The next sections deal with the combinatorial interpretation of Theorem 2.1 from Section 2. In Section 3 a formula for the Kostka-Green-Foulkes polynomials corresponding to bitableaux is presented. As a corollary, we deduce a new rule to compute the Clebsch-Gordan numbers, i.e. the multiplicities of irreducible components V_λ in the tensor product $V_\mu \otimes V_\nu$ (Example 8, Section 3). In example 5, section 3, we derive certain inequalities between the Kostka polynomials [11]. In example 5, section 3, we derive certain inequalities between the Kostka polynomials. In a particular case more general inequalities were derived by A. Lascoux. In example 7 we derive a stability theorem for configurations. Note that the problem of computation of the generalized exponents [23] for $sl(n)$ may be reduced to the computation of some special Kostka polynomials [12], [13], [14], and so Theorem 3.1 implies an explicit formula for these exponents. In Section 4 we present a proof of Theorem 3.1 (section 3) in an important special case. Following [10], we establish a bijective correspondence between the rigged configurations of type (λ, μ) and the set of the standard tableaux of shape λ and weight μ. The formula for the Kostka polynomials is deduced from the properties of this correspondence. At the end of Section 4 we discuss a relation between the involution on the set of the standard tableaux induced by the «inversion of the quantum numbers» and the Schützenberger involution on the set of the standard tableaux induced by the involution. We prove that the latter maps the charge functional into the index.

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1. THE YOUNG BITABLEAUX

In this section we recall the basic definitions and the necessary facts concerning the combinatorics of Young tableaux used in the present paper. The proofs and details may be found in [11], [10].

1.1. Partitions, Young diagrams, tableaux and supertableaux

Let $\lambda = (\lambda_1, \lambda_2, \ldots)$ be a sequence of integers with finitely many nonzero terms, and define

$$|\lambda| = \lambda_1 + \lambda_2 + \ldots \tag{1.1}$$

Recall that λ is a partition of n if the sequence is nonnegative, weakly decreasing, and $|\lambda| = n$. Nonzero terms in (1.1) are called the parts of the partition λ and their number is called the length of the partition λ, denoted by $l(\lambda)$. The diagram of the partition λ is the set

$$\mathcal{D}_\lambda = \{(i,j) \in \mathbb{Z}^2 | 1 \le j \le \lambda_i\}, \tag{1.2}$$

which may be viewed as a collection of rectangular unit boxes in the plane with the centers lying at the points $(i,j) \in \mathbb{Z}^2$, $1 \le j \le \lambda_i$. Partitions are partially ordered by inclusion of diagrams; we define

$$\lambda \subseteq \mu \qquad \text{iff} \qquad \mathcal{D}_\lambda \subseteq \mathcal{D}_\mu. \tag{1.3}$$

The set $\mathcal{D}_\lambda$ and its picture on the plane will be called the Young diagram of the shape λ. If λ and ν are partitions and $\nu \subseteq \lambda$, let us call the set $\mathcal{D}_\lambda \backslash \mathcal{D}_\nu$ the Young diagram of the shape $\lambda\backslash\nu$. The conjugate λ' is the partition whose diagram is the transpose of $\mathcal{D}_\lambda$; thus, there are λ'_i boxes in the i-th column of $\mathcal{D}_\lambda$:

$$\lambda'_i = \text{Card } |\{j|\lambda_j \geq i\}|. \tag{1.4}$$

It is clear that

$$m_i(\lambda) := \text{Card } |\{j|\lambda_j = i\}| = \lambda'_i - \lambda'_{i+1}. \tag{1.5}$$

For every partition λ we put

$$\begin{aligned} n(\lambda) &= \sum_{i\geq 1}(i-1)\lambda_i = \sum_{i\geq 1}\binom{\lambda'_i}{2}, \\ Q_n(\lambda) &= \sum_{j\geq 1}\min(n,\lambda_j) = \sum_{j=1}^{n}\lambda'_j. \end{aligned} \tag{1.6}$$

On the set of partitions we define the following relation of order: $\lambda \geq \mu$ iff

$$\sum_{i=1}^{k}\lambda_i \geq \sum_{i=1}^{k}\mu_i \qquad \text{for all } k.$$

A composition λ of n is a collection $\lambda_j \in \mathbf{Z}_+$ such that $|\lambda| := \sum \lambda_j = n$. It is clear that the functionals $l(\lambda), \lambda'_i, m_i(\lambda), Q_n(\lambda)$ may be defined not only for partitions but also for compositions. For example,

$$Q_n(\lambda) = \sum_{j\geq 1}\min(n,\lambda_j).$$

Let X be a totally ordered set. A Young tableau of shape $\lambda\backslash\nu$ on the set X is an assignement $T : \mathcal{D}_{\lambda\backslash\nu} \to X$ of elements of the set X to the boxes in the diagram of shape $\lambda\backslash\nu$ which satisfies the following condition.

The rows and columns are weakly increasing; i.e. $T(i,j) \leq T(i,j+1)$ and $T(i,j) \leq T(i+1,j)$ whenever $(i,j),(i,j+1),(i+1,j) \in \mathcal{D}_{\lambda\backslash\nu}$.

For $x \in X$ let us denote by T_x the inverse image $T^{-1}(x)$ of the element x in the tableau T. The set T_x is called a horizontal (resp. vertical) strip, if there is at most one box in every column (resp. line).

A standard tableau T on the set X is a horizontal strip for all $x \in X$. A bitableau on the set X is by definition a tableau such that for all $x \in X$ the set T_x is either a horizontal or a vertical strip.par For the future we assume that

$$X = X_{m,n} = \{1 < 2 < \ldots < \overline{1} < \overline{2} < \ldots < \overline{n}\},$$

and also that every set $T_j, j \in X$ is a horizontal strip, and every set $T_{\bar{j}}, \bar{j} \in X$ is a vertical strip.

The pair $(\mu|\eta)$ where $\mu = (\mu_i | i \in X), \eta = (\mu_{\bar{j}} | j \in X)$ is called the weight of bitableau T. Let us denote by $SBY(\lambda\backslash\nu, \mu|\eta)$ the set of bitableaux of the shape $\lambda\backslash\nu$ and the weight $(\mu|\eta)$. Set $STY(\lambda\backslash\nu, \mu) = SBY(\lambda\backslash\nu, \mu|0)$.

1.2. Schur, Hall-Littlewood functions and Kostka-Green-Foulkes polynomials

Let $\nu \subseteq \lambda$ be partitions, and $x = (x_1, \dots, x_N)$ be a collection of variables, $l(\lambda) \le N$.

The Schur function $S_{\lambda\backslash\nu}(x)$ corresponding to the skew Young diagram $\lambda\backslash\nu$ is defined by the following formula

$$S_{\lambda\backslash\nu}(x) := \sum_T x^{\mu(T)}, \tag{1.7}$$

where $X^{\mu(T)} = x_1^{\mu_1} \dots X x_N^{\mu_N}, \mu := \mu(T)$ is the weight of the tableau $T \in STY(\lambda, \mu)$, $l(\mu) \le N$.

For the Schur function $S_{\lambda\backslash\nu}(x)$ there is the following representation

$$S_{\lambda\backslash\nu}(x) = \sum_\mu K_{\lambda\backslash\nu,\mu} \cdot m_\mu(x),$$

where the summation ranges over the partitions μ of $|\lambda| - |\nu|$ into at most N parts, the numbers $K_{\lambda\backslash\nu,\mu}$ are equal to Card $|STY(\lambda\backslash\nu, \mu)|$, and $m_\mu(x)$ is the monomial symmetric function corresponding to the partition μ:

$$m_\mu(x) = \sum_{w \in S_N} w(x_1^{\mu_1} \dots x_N^{\mu_N}).$$

Let $\lambda \supseteq \nu$ be partitions, $x = (x_1, \dots, x_N)$, $y = (y_1, \dots, y_M)$ be two collections of variables, $l(\lambda) \le M + N$.

The Schur superfunction $HS_{\lambda\backslash\nu}(x|y)$ corresponding to the skew Young diagram $\lambda\backslash\nu$ is defined by the following formula [18], [17], [19]:

$$HS_{\lambda\backslash\nu}(x|y) = \sum_T x^{\mu(T)} y^{\eta(T)}, \tag{1.8}$$

where $(\mu(T)|\eta(T))$ is the weight of bitableau T, $T \in SBY(\lambda\backslash\nu, \mu|\eta), l(\mu) \le N, l(\eta) \le M$.

For the Schur superfunction $HS_{\lambda\backslash\nu}(x|y)$ there is the following representation

$$HS_{\lambda\backslash\nu}(x|y) = \sum SK_{\lambda\backslash\nu,\mu|\eta} \cdot m_\mu(x) m_\eta(y), \tag{1.9}$$

where the summation ranges over the partitions μ, η such that $l(\mu) \leq N, l(\eta) \leq M, |\mu| + |\eta| = |\lambda| - |\nu|$, the numbers $SK_{\lambda\backslash\nu,\mu|\eta}$ are equal to Card $|SBY(\lambda\backslash\nu, \mu|\eta)|$.

The Hall-Littlewood function $P_\lambda(x;q)$ corresponding to the partition λ is defined by the following formula

$$P_\lambda(x;q) = [v_{N-l(\lambda)}(q)v_\lambda(q)]^{-1} \sum_{\sigma \in S_N} \sigma \left(x_1^{\lambda_1} \ldots x_N^{\lambda N} \prod_{i<j} \frac{x_i - qx_j}{x_i - x_j} \right), \tag{1.10}$$

where $v_\lambda(q) := \prod_{i \geq 1} v_{m_i}(q), m_i = \lambda_i' - \lambda_{i+1}'$ and the polynomials $v_m(q)$ are equal to $\pi_{i=1}^m \frac{1-q^i}{1-q}$.

Let us define the Kostka-Green-Foulkes polynomials $K_{\lambda,\mu}(q) \in \mathbf{Z}[q]$ by the decomposition

$$S_\lambda(x) = \sum_\mu K_{\lambda,\mu}(q) P_\mu(x;q). \tag{1.11}$$

Now we define the Kostka polynomials for skew diagrams and supertableaux.

Let λ, μ, ν, η be some partitions.

We define the Kostka polynomials $K_{\lambda\backslash\nu,\mu}(q)$, $K_{\lambda\backslash\nu,\mu|\eta}(q)$ by the following decompositions

$$S_{\lambda\backslash\nu}(x) = \sum_\mu K_{\lambda\backslash\nu,\mu}(q) P_\mu(x,q), \tag{1.12}$$

$$HS_{\lambda\backslash\nu}(x|y) = \sum_{\mu,\eta} K_{\lambda\backslash\nu,\mu|\eta}(q) P_\mu(x,q) P_\eta(y,q). \tag{1.13}$$

1.3. The charge of a tableau

Let μ, η be two partitions. Following Lascoux and Schützenberger we define the charge $C(T)$ of a bitableau $T \in SBY(\lambda\backslash\nu, \mu|\eta)$. Consider a word $\omega = a_1 \ldots a_N$ with positive integer elements a_i. The weight of ω is the sequence $\mu = (\mu_1, \mu_2, \ldots)$ where μ_i is the number of those a_j's which are equal to i. Suppose that μ is a partition.

(i) We first assume that ω is a standard word, i.e., that its weight is $\mu = (1^N)$. Let us index all elements of ω as follows: the index of 1 is equal to 0, and if the index of K is i, the index of $K+1$ is either i or $i+1$ according to the location of $K+1$ either to the right or to the left of K. The charge $C(\omega)$ of ω is then the sum of all its indices.

(ii) Assume now that ω is a word of weight μ and μ is a partition. We extract a standard subword out of ω in the following way. Reading ω from left to right we choose the first entry of 1, then the first entry of 2 to the right of the 1 chosen, and so on. If at some step there is no $S+1$ to the right of the S chosen before, we come back to the beginning of the word. This operation extracts from ω a standard subword ω_1 out of ω. Let us now delete the word ω_1 from ω and repeat the operation, thus obtaining ω_2, etc.

The charge of ω is defined to be the sum of the charges of the standard subwords obtained in this way: $C(\omega) = \sum c(\omega_i)$. We note that the charge of ω is zero if and only if ω is a lattice word.

(iii) Let now $T \in STY(\lambda\backslash\nu, \mu)$. Reading successively the rows of T from left to right starting from the top, we obtain a word $\omega(T)$. The charge $C(T)$ of T is by definition $C(\omega(T))$.

(iv) Let $T \in SBY(\lambda\backslash\nu, \mu|\eta)$ where μ and η are partitions, $l(\mu) = m, l(\eta) = n$. Associated with the bitableau T are two tableaux $T(\overline{0})$ and $T(\overline{1})$ composed of the boxes of T filled with the integers $\{1, \dots, m\}$ and $\{\overline{1}, \dots, \overline{n}\}$, respectively. We set

$$C_{\text{ev}}(T) = C(T(\overline{0})), C_{\text{odd}}(T) = C(T(\overline{1})'),$$
$$C(T) = C_{\text{ev}}(T) + C_{\text{odd}}(T),$$

where T' is the conjugate of T.

The pair $(C_{\text{ev}}(T), C_{\text{odd}}(T))$ is called the charge of the supertableau T, and $C(T)$ is its total charge.

1.4. The index of a tableaux

Let ω be a standard word of length N. By $\mathcal{D}(\omega)$ we denote the set of integers j such that j and $j+1$ are contained in ω and $j+1$ stands to the right of j in ω. We set

$$d = (\omega) = \text{Card } |\mathcal{D}(\omega)|, \text{des}(\omega) = \sum_{j \in \mathcal{D}(\omega)} j.$$

Clearly, $C(\omega) = \binom{N}{2} - d \cdot N + \text{des}(\omega)$. We define the index of ω by $\text{Ind}(\omega) = \binom{N}{2} - \text{des}(\omega)$. Set $d(T) = d(\omega(T)), \text{Ind}(T) = \text{Ind}(\omega(T))$, where $\omega(T)$ is defined by (1.3), iii).

1.5. The charge functional and Kostka polynomials

Theorem (Lascoux-Schützenberger [21], [11]).

$$K_{\lambda,\mu}(q) = \sum_{T \in STY(\lambda,\mu)} q^{C(T)}. \tag{1.14}$$

Theorem (Thomas [20]).

$$K_{\lambda,(1^N)}(q) = \sum_{T \in STY(\lambda)} q^{\mathrm{Ind}(T)}. \tag{1.15}$$

It follows that

$$\sum_{T \in STY(\lambda,\mu)} q^{C(T)} = \sum_{T \in STY(\lambda)} q^{\mathrm{Ind}(T)}. \tag{1.16}$$

In 4, Corollary 4.2, we show that Schützenberger's involution S on the set $STY(\lambda)$ carries the charge functional into the index: $C(T) = \mathrm{Ind}(S(T))$. In this way we get a combinatorial proof of the identity (1.16).

For the Kostka polynomials associated with skew diagrams and supertableaux (see (1.12) and (1.13)), we have the following analog of (1.14).

THEOREM 1.1. We have

$$K_{\lambda\backslash\nu,\mu}(q) = \sum_{T \in STY(\lambda\backslash\nu,\mu)} q^{C(T)}, \tag{1.17}$$

$$K_{\lambda\backslash\nu,\mu|\eta}(q) = \sum_{T \in STY(\lambda\backslash\nu,\mu|\eta)} q^{C(T)}. \tag{1.18}$$

Formula (1.17) follows from (1.14) and the fact that the bijection (see [11], §9)

$$STY(\lambda\backslash\nu) \simeq \coprod_{\rho} STY_0(\lambda\backslash\nu,\rho) \times STY(\rho,\mu)$$

is compatible with the charge functional: if $T \in STY(\lambda\backslash\nu,\rho) \times STY(\rho,\mu)$ then $C(T) = C(\pi)$. Recall that $STY_0(\lambda\backslash\nu,\rho)$ consists of the standard tableaux T of shape $\lambda\backslash\nu$, weight μ and charge zero.

1.6. Schützenberger's involution

Let $T \in STY(\lambda,\mu)$, let μ be a composition, $\mu = (\mu_1,\dots,\mu_l)$ all $\mu_i \neq 0$. Eliminate the integer $T(1,1)$ from T. Using «jeu de taquin» following Schützenberger [16] we derive a standard tableau T' which has one box less than T. Insert $T(1,1)$ into the free box. By repeating this operation we come down to a tableau $\widetilde{T}$ of shape λ such that

$$\widetilde{T}(i,j) \geq \widetilde{T}(i,j+1), \quad \widetilde{T}(i,j) > \widetilde{T}(i+1,j).$$

Let ST denote the tableau such that $ST(i,j) >= l + 1 - \widetilde{T}(i,j)$. It is clear that $ST \in STY(\lambda, \overleftarrow{\mu})$ where $\overleftarrow{\mu} = (\mu_l,\mu_{l-1},\dots,\mu_1)$. Therefore there is a mapping

$$S : STY(\lambda,\mu) \to STY(\lambda,\overleftarrow{\mu}). \tag{1.19}$$

The mapping S is called Schützenberger's involution if the weight $\mu = (1^N)$.

2. DECOMPOSITION OF TENSOR PRODUCTS OF IRREDUCIBLE REPRESENTATION OF $GL(N)$, [7]

Let $\lambda, \mu^{(1)}, \mu^{(2)}, \dots$ be Young diagrams, $|\lambda| = \sum_{j\geq 1} |\mu^{(j)}|$.

A set $\{\nu\}$ consisting of Young diagrams $\nu^{(k)}, k = 0, 1, \dots$ such that

$$|\nu^{(k)}| = \sum_{l\geq k+1} \left(\lambda_l - \sum_{j\geq 1} \mu_l^{(j)}\right) = \sum_{l\leq k} \left(\sum_{j\geq 1} \mu_l^{(j)} - \lambda_l\right) \tag{2.1}$$

is called a configuration of type $(\lambda, \{\mu\})$.

The number

$$\begin{aligned} P_n^{(k)}(\nu) &= P_n^{(k)}(\{\nu\}, \{\mu\}) = \\ &= \sum_{j\geq 1} \min(n, \mu_k^{(j)} - \mu_{k+1}^{(j)}) + Q_n(\nu^{(k-1)}) - \\ &\quad -2Q_n(\nu^{(k)}) + Q_n(\nu^{(k+1)}) \end{aligned} \tag{2.2}$$

is called the number of vacancies for the configuration $\{\nu\}$. A configuration of type $(\lambda, \{\mu\})$ is called admissible if $P^{(k)}(\nu) \geq 0$ for all $n, k \geq 1$.

For a collection $\lambda, \{\mu^{(j)}\})$ of Young diagrams put

$$Z(\lambda|\{\mu\}) = \sum_{\{\nu\}} \prod_{k\geq 1} \prod_n \binom{P_n^{(k)} + m_n(\nu^{(k)})}{m_n(\nu_{(k)})}, \tag{2.3}$$

where $m_n(\nu^{(k)})$ is the number of rows of length n in the diagram $\nu^{(k)}$. The summation in (2.3) ranges over all admissible type $(\lambda, \{\mu\})$ configurations and

$$\binom{n}{m} = \begin{cases} 0 & , \text{if } m \notin [0, n], \\ \dfrac{n!}{m!(n-M)!} & , \text{if } 0 \leq m \leq n \end{cases}$$

is the binomial coefficient.

THEOREM 2.1. *Let $\lambda, \{\mu^{(j)}\}$ be a collection of Young diagrams such that $l(\lambda) \leq N, l(\mu_{(j)} \leq N$ and $|\lambda| = \sum_{j\geq 1} \mu^{(j)}$.*

Then the multiplicity of the irreducible component V_λ in $\bigotimes_i V_{\mu^{(i)}}$ is estimated as follows

$$\text{mult}_{V_\lambda}(\bigotimes_i V_{\mu^{(i)}}) \leq Z(\lambda|\{\mu\}). \tag{2.4}$$ ■

It is not difficult to give examples showing that the inequality in (2.4) may be strict. On the other hand, in Theorem 2.2 we describe a class of diagrams $\{\mu^{(i)}\}$ for which an exact equality in (2.4) holds.

THEOREM 2.2. *Assume that $l(\lambda) \le N$ and all Young diagrams $\{\mu^{(i)}\}$ are rectangular, i.e. have the form $(m^l), l \le N$. Then*

$$\text{mult}_{V_\lambda}(\bigotimes_i V_{\mu^{(i)}}) = Z(\lambda|\{\mu\}). \tag{2.5}$$ ∎

Theorem 2.2 (in an equivalent form) is proved analytically in [7] with the help of generating functions and the multidimensional residues.

Remarks

1. It seems plausible that conversely if the equality in (2.4) holds for all λ, then all the diagrams $\mu^{(i)}$ are rectangular.

2. An essential point of the proof of Theorem 2.2 is to check the following identity (which makes sense under the assumptions of Theorem 2.2)

$$\sum_{\{\nu\}} \prod_{k\ge 1} \prod_n (-1)^{m_n(\nu^{(k)})} \binom{-P_k^{(k)}(\nu) - 1}{m_n(\nu^{(k)})} = 0, \tag{2.6}$$

where the summation ranges over all possible type $(\lambda, \{\mu\})$ configurations such that

$$P_n^{(k)}(\nu) + m_n(\nu^{(k)}) < 0.$$

3. The function $Z(\lambda|\{\mu\})$ has a simple meaning in the quantum inverse scattering method. If all Young diagrams $\mu^{(i)}$ are rectangular, $Z(\lambda|\{\mu\})$ is equal to the number of Bethe's vectors in the generalized Heisenberg ferromagnet model on the 1-dimensional lattice with spins $\mu^{(1)}, \mu^{(2)}, \ldots$, which transform according to the representation V_λ. Identity (2.6) means simply that interacting positive spins on the lattice cannot create excitations with negative spins. Theorem 2.2. is equivalent to the completeness of the multiplet family generated by the Bethe's vectors in the generalized Heisenberg model. Representations corresponding to rectangular Young diagrams are distinguished from all the others by the fact that in this case Bethe's vectors form an orthogonal system.

4. In formula (2.2) we have $\nu^{(0)} = 0$, as is evident from (2.1). However, for some special choice of the diagrams $\mu^{(1)}$ it is convenient to use a different boundary condition for $\nu^{(0)}$ (cf. (2.9), (3.9)).

5. Both the left hand side and the right hand side of (2.5) admit a natural combinatorial interpretation. The former describes a certain class (depending on the concrete choice of $\mu^{(i)}$) of λ shaped tableaux, while the latter may be described in terms of rigged configurations (see (4.1)). Theorem 2.2 asserts that both sets have the same number of elements. An explicit bijection between them is constructed in [9] when all $\mu^{(i)} = (1)$ and in [10] when $\mu^{(i)} = (m_i)$ or $\mu^{(i)} = (1^{n_i})$. The general case will be described by the author in a separate publication.

6. Theorem 2.2 implies a new formula for the multiplicity of a weight $(\mu|\eta)$ in the irreducible tensor $GL(N|M)$-module [10].

THEOREM 2.3. *The following equality holds*

$$\dim V\lambda(\mu|\eta) = \sum_{\{\nu\}} \prod_{k\geq 1} \prod_{h} \binom{P_n^{(k)}(\nu) + m_n(\nu^{(k)})}{m_n(\nu^{(k)})}, \tag{2.8}$$

where the summation in (2.8) ranges over all sets of Young diagrams $\{\nu^{(k)}\}$ *such that*

(i) $|\nu^{(k)}| = \sum_{j\geq k+1} (\lambda_j - \eta'_j),$

(ii) $P_n^{(k)}(\nu) \geq 0$

for all $n, k \geq 1$ *, where*

$$P_n^{(k)}(\nu) = \eta'_k - \eta'_{k+1} + Q_n(\nu^{(k-1)}) - 2Q_n(\nu^{(k)}) + Q_n(\nu^{(k+1)}), \tag{2.9}$$
$$\nu^{(0)} := \mu.$$

■

3. THE KOSTKA-GREEN-FOULKES POLYNOMYALS

In this section we give a formula for calculating the Kostka-Green-Foulkes polynomials corresponding to supertableaux (see (1.13)) and deduce some consequences from it: inequalities between the Kostka-polynomials (example 5), a new rule for finding the Clebsch-Gordan numbers (example 7). The proof of the main result of this section, Theorem 3.1, will be given in section 4 for the case of Kostka polynomials corresponding to tableaux.

Let λ, ρ, μ, η be some partitions and ρ correspond to a rectangular Young diagram $\rho = (m^l)$.

THEOREM 3.1. *The following equality holds*

$$k_{\lambda\backslash\rho,\mu|\eta}(q) = \sum_{\{\alpha\}} q^{C(\{\alpha\})} \prod_{k\geq 1} \prod n \begin{bmatrix} P_n^{(k)}(\alpha) + \alpha_n^{(k)} - \alpha_{n+1}^{(k)} \\ \alpha_n^{(k)} - \alpha_{n+1}^{(k)} \end{bmatrix}_q, \tag{3.1}$$

where the summation ranges over all sets of Young diagrams $\{\alpha^{(k)}\}$ *such that*

(i) $|\alpha^{(k)}| = \sum_{j\geq k+1} (\lambda_j - \rho_j - \eta'_j),$

(ii) $P_n^{(k)}(\alpha) \geq 0$

for all $n, k \geq 1$ *, where*

$$P_n^{(k)}(\alpha) := \eta_k' - \eta_{k+1}' + \min(n, \rho_k - \rho_{k+1}) + \sum_{j \leq n}(\alpha_j^{(k-1)} - 2\alpha_j^{(k)} + \alpha_j^{(k+1)}), \tag{3.2}$$

$$\alpha^{(0)} := \mu',$$

and $\begin{bmatrix} m \\ n \end{bmatrix}_q$ *is the* q*-binomial Gauss coefficient:*

$$\begin{bmatrix} m \\ n \end{bmatrix}_q = \prod_{j=1}^{n} \frac{1 - q^{m-j+1}}{i - q^j}$$

$$\text{if} \quad 0 \leq n \leq m, \begin{bmatrix} m \\ n \end{bmatrix}_q = 0, \quad \text{if} \quad n \notin [0, m]$$

Exponent $C(\{\alpha\})$ *in* (3.1) *is equal to*

$$C(\{\alpha\}) = \sum_{k \geq 0} \sum_n \binom{\alpha_n^{(k)} - \alpha_n^{(K+1)}}{2}, \tag{3.3}$$

where $\binom{\beta}{2} := \dfrac{\beta(\beta - 1)}{2}, \beta \in \mathbf{R}$. ∎

Remarks 3.1. Let us set $\nu^{(k)} = (\alpha^{(k)})'$. The vacancies numbers $P^{(k)}(\alpha)$ (see (3.2)) take the following form in terms of diagrams $\{\nu^{(k)}\}$:

$$P_n^{(k)}(\alpha) = P_n^{(k)}(\nu) = \eta_k' - \eta_{k+1}' + \min(n, \rho_k - \rho_{k+1}) + Q_n(\nu^{(k-1)}) - 2Q_n(\nu^{(k)}) + Q_n(\nu^{(k+1)}), \quad \nu^{(0)} := \mu. \tag{3.2a}$$

Note that $m_n(\nu^{(k)}) = \alpha_n^{(k)} - \alpha_{n+1}^{(k)}$, so the formulas (3.1) and (2.3) are equivalent. In the sequel the collections of diagrams $\{\nu\}$ (or $\{\alpha\}$) which satisfy the condition (i) will be called $(\lambda \backslash \rho)$-configurations, and those satisfing (i) and (ii) $(\mu|\eta)$-admissible.

Now we give a few examples using Theorem 3.1.

Example 1. Let $\lambda = (n, 1^{m-1}), \mu$ be partitions, $|\lambda| = |\mu|, \rho = (0), \eta = (0)$. Then we have

$$K_{\lambda,\mu}(q) = q^C \begin{bmatrix} \mu_1' - 1 \\ \lambda_1' - 1 \end{bmatrix}_q, \tag{3.4}$$

where $C = n(\mu) - (\lambda_1' - 1)\mu_1' + n(\lambda)$.

Indeed, there is only one admissible configuration $\{\alpha\}$ which consists of the diagrams $\alpha^{(k)} = (m - k), 1 \leq k \leq m - 1$. The vacancies numbers are equal to

$$P_1^{(1)}(\alpha) = \mu_1' - m, \; P_1^{(k)}(\alpha) = 0, \; 2 \leq k \leq m - 1.$$

The charge $C(\{\alpha\})$ may be found from (3.3). Note that $\lambda \geq \mu \equiv \mu_1' \geq \lambda_1' = m$.

Example 2. Let $\lambda = (n, 2, 1^m), \mu$ be partitions, $|\mu| = |\lambda|, \rho(0), \eta = (0)$. Then we have

$$K_{\lambda,\mu}(q) = q^{C_1} \begin{bmatrix} \lambda_1' - 1 \\ 1 \end{bmatrix} \begin{bmatrix} \mu_1' - \lambda_2 \\ \lambda_1' \end{bmatrix} + \qquad (3.5)$$
$$+ q^{C_2} \begin{bmatrix} \mu_1' + \mu_2' - \lambda_1 - 1 \\ 1 \end{bmatrix} \begin{bmatrix} \mu_1' - \lambda_2 \\ \lambda_1' - 2 \end{bmatrix},$$

where $c_1 = n(\mu) + n(\lambda) - \lambda_1'(\mu_1' - 1), c_2 = n(\mu) + n(\lambda) - \mu_1'(\lambda_1' - 1) - \mu_2'$.

In fact, there are two admissible configurations

$$\{\alpha\} = \{\alpha^{(k)} = (n_k)\},$$
$$\{\alpha\} = \{\alpha^{(1)} = (n_1 - 1, 1), \alpha^{(k)} = (n_k), k \geq 2\}.$$

For the first configuration $P_1^1(\alpha) = \mu_1' - 2n_1 + n_2 = \mu_1' + \lambda_1 - \lambda_2 - |\lambda|, P_1^{(2)}(\alpha) = \lambda_2 - \lambda_3 - 1, P_1^{(k)} = \lambda_k - \lambda_{k+1}, k \geq 3$. Formula (3.5) is proved.

Example 3. Let λ, μ be some partitions of the number $n, \lambda \geq \mu$. Let us define the configuration $\{\alpha\}$ in the following way: $\alpha_n^{(k)} := \max(\lambda_n' - k, 0), k \geq 1$. It is clear that $\sum_n \alpha_n^{(k)} = \sum_{j \geq k+1} \lambda_j$. We assert that $\{\alpha\}$ is an admissible configuration. Indeed, we have $P_m^{(1)}(\alpha) = 0$, if $m \leq \lambda_k, k \geq 2$. On the other hand, it is well-known (see [11]) that $\lambda \geq \mu \equiv Q_m(\mu) \geq Q_m(\lambda)$.

The contribution to the Kostka polynomial $K_{\lambda,\mu}(q)$ from the configuration $\{\alpha\}$ is equal to

$$R(q) = q_C \prod_{m=1}^{\lambda_2} \begin{bmatrix} Q_m(\mu) - Q_m(\lambda) + \lambda_m' - \lambda_{m+1}' \\ \lambda_m' - \lambda_{m+1}' \end{bmatrix}_q . \qquad (3.6)$$

Note that $\deg r(q) = n(\mu) - n(\lambda) = \deg K_{\lambda,\mu}(q)$.

For each pair of diagrams λ, μ let us define the numbers $a(\lambda, \mu)$ and $b(\lambda, \mu)$ in the following way

$$K_{\lambda,\mu}(q) = b(\lambda, \mu) q^{a(\lambda,\mu)} (1 + O(q)). \qquad (3.7)$$

CONJECTURE. (I.G. Macdonald [11]) $a(\lambda, \mu) = a(\mu', \lambda')$.

It seems a very interesting task to study the behaviour of the number $b(\lambda, \mu)$. Is it bounded when $|\lambda| = |\mu| \to \infty$? What values may take the numbers $b(\lambda) = \max\{b(\lambda, \mu) \mid |\mu| = |\lambda|\}, b_N = \max\{b(\lambda) \mid |\lambda| = N\}$? I don't know any diagrams λ, μ such that $b(\lambda, \mu) \geq 4$.

Let us formulate a few conjectures about the numbers $b(\lambda, \mu)$:

1. (see [10]) $b(\lambda, \mu) = b(\mu', \lambda)$.
2. $b_N \to \infty$, when $N \to \infty$.
3. $b(\lambda, \mu) = 2$ iff $\lambda_1' = \mu_1' - \mu_2' \geq 2, \mu_1' - \lambda_1' \geq 2, \mu_1' + \mu_2' - \lambda_1' \geq 2$.

Example 4. Let be $\lambda = (5,3,2,1^N)$, $\mu = (3,3,2,1^{N+2})$. There are four admissible configurations, and

(3.8)
$$\begin{aligned} K_{\lambda,\mu}(q) =& q^3 \begin{bmatrix} 3 \\ 1 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \end{bmatrix} \begin{bmatrix} \lambda_1' - 1 \\ 2 \end{bmatrix} + q^3 \begin{bmatrix} 3 \\ 1 \end{bmatrix} \begin{bmatrix} \lambda_1' - 2 \\ 1 \end{bmatrix} + \\ &+ q^3 \begin{bmatrix} 2 \\ 1 \end{bmatrix} \begin{bmatrix} \lambda_1' - 2 \\ 1 \end{bmatrix} + q^5 \begin{bmatrix} 2 \\ 1 \end{bmatrix} \begin{bmatrix} \lambda_1' - 2 \\ 1 \end{bmatrix}. \end{aligned}$$

It is clear that $\mu' = (N+5,3,2)$, $\lambda' = (N+3,3,2,1,1)$ and $K_{\mu',\lambda'}(q) = q^3 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + q^3 \begin{bmatrix} 3 \\ 1 \end{bmatrix} + q^3 \begin{bmatrix} 3 \\ 1 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \end{bmatrix}$. Therefore

$$a(\lambda,\mu) = a(\mu',\lambda') = 3, b(\lambda,\mu) = b(\mu',\lambda') = 3.$$

Example 5. Inequalities between the Kostka polynomials. Let λ, μ be partitions, $\rho = (m^l)$, $\rho \subseteq \lambda$. We define

(.3.9)
$$\overline{K}_{\lambda\backslash\rho,\mu}(q) = q^{n(\mu)-n(\lambda\backslash\rho,\mu)} K_{\lambda\backslash\rho,\mu}(q^{-1}).$$

It is clear that $\overline{K}_{\lambda\backslash\rho,\mu}(q)$ is a polinomial, $\overline{K}_{\lambda\backslash\rho,\mu}(0) = 1$, because $\deg \overline{K}_{\lambda\backslash\rho,\mu}(q) = n(\mu) - n(\lambda\backslash\rho)$ and its leading coefficient is equal to 1. It follows from Theorem 3.1 that polynomial $\overline{K}_{\lambda\backslash\rho,\mu}(q)$ has a similar expression in which the exponent $\overline{C}(\{\alpha\})$ is calculated by the following formula

$$\begin{aligned} \overline{C}(\{\alpha\}) =& n(\mu) - n(\lambda\backslash\rho) - C(\{\alpha\}) - \\ &- \sum_{k\geq 1} \sum_n [\alpha_n^{(k)} - \alpha_{n+1}^{(k)}] P_n^{(k)}(\alpha). \end{aligned}$$

LEMMA 3.1. *We have*

(3.10)
$$\begin{aligned} \overline{C}(\{\alpha\}) =& C(\{\alpha\}) + \sum \mu_n' \cdot \alpha_n^{(1)} - n(\mu) - n(\lambda\backslash\rho) = \\ =& \sum_{k\geq 1} \sum_n \binom{\alpha_n^{(k)} - \alpha_n^{(k+1)}}{2} \sum_{k\geq 1} \binom{\alpha_n^{(1)} + 1}{2} - n(\lambda\backslash\rho). \end{aligned}$$

The proof of Lemma 3.1 follows from the equality

$$\begin{aligned} \sum_{k\geq 1} \sum_n [\alpha_n^{(k)}] P_n^{(k)}(\alpha) =& \sum_{n\geq 1} \alpha_n^{(0)} \alpha_n^{(1)} - \\ & -2 \sum_{k\geq 1} \sum_n \alpha_n^{(k)} (\alpha_n^{(1)} - \alpha_n^{(1)}). \end{aligned}$$ ∎

Note that the exponent $\overline{C}(\{\alpha\})$ (see (3.10)) depends only on a configuration $\{\alpha\}$ and does't depends on weight μ.

THEOREM 3.2. *If* $\mu_1 \le \mu_2$, *then*

$$\overline{K}_{\lambda\backslash\rho,\mu_2}(q) \le \overline{K}_{\lambda\backslash\rho,\mu_1}(q). \tag{3.11}$$

For beginning let us remark that

$$P_n^{(k)}(\{\alpha\},\mu_1)P_n^{(k)}(\{\alpha\},\mu_2) = \delta_{k,1}(Q_n(\mu_1') - Q_n(\mu_2')) \ge 0,$$

because $\mu_1' \ge \mu_2'$. Therefore every μ_2-admissible configuration is also μ_1-admissible. The proof of Theorem 3.1 follows easily from well-known result:

$$\text{if} \qquad m_1 \le m_2 \qquad \text{then} \qquad \begin{bmatrix} m_1 \\ n \end{bmatrix}_q \le \begin{bmatrix} m_2 \\ n \end{bmatrix}_q .$$

In the case $\rho = 0$ the Theorem 3.1 is proved also by A . Lascoux (private communication). ∎

Example 6. Take $\lambda = (\alpha_1 + N, \ldots, \alpha_r + N, \underbrace{N, N, \ldots, N}_{n-r-s}, N - \beta_S, \ldots N - \beta_1) := (\alpha, -\beta)_n + (N^n), \eta = ((n-1)^N), |\beta| = N, l(\alpha) + l(\beta) \le n.$

LEMMA 3.2. *The following equality holds*

$$K_{\lambda,\mu|\eta}(q) = q^{n(\beta)} K_{\alpha,\mu}(q) k_{\beta,(1^N)}(q). \tag{3.12}$$

To avoid confusion, denote configurations that enter into (3.1) by $\{\nu\}$ instead of $\{\alpha\}$. Consider a $(\lambda, \mu|\eta)$-configuration $\{\nu\}$ see (2.11)). Then we have

$$|\nu^{(k)}| = \sum_{j \ge k+1} \alpha_j,\ 1 \le k \le n-s,\ |\nu^{(n-s)}| = 0,$$

$$|\nu^{(k)}| = \sum_{j \ge n-s+1} \beta_j, n-s+1 \le k \le n-1.$$

Consequently, we may consider (α, μ)-configuration $\{\nu_\alpha^{(k)} := \nu^{(k)} | 1 \le k \le n-s\}$ and β-configuration $\{\nu_\beta^{(k)} := \nu^{(k)} | n-s+1 \le k \le n-1\}$. It is clear that

$$P_m^{(k)}(\nu) = P_m^{(k)}(\nu_\alpha), 1 \le k \le n-s-1, \nu_\alpha^{(0)} := \mu,$$

$$P_m^{(k)}(\nu) = P_m^{(n-k)}(\nu_\beta), n-s+1 \le k \le n-1, \nu_\beta^{(n)} := (1^N).$$

From (3.3) and (3.10) it follows that

$$C(\{\nu\}) = C(\{\nu_\alpha\} + C(\{\nu_\beta\} - \binom{N}{2} + N \cdot \nu_{\beta,1}^{(1)} =$$

$$= C(\{\nu_\alpha\} + \overline{C(\{\nu_\beta\}} + n(\beta).$$

Lemma 3.2 is proved. ∎

Example 7. Take $\alpha = \beta = (2,1), \lambda = (4,3,2^{n-4},1) = (\alpha,-\beta)_n + (2^n), \mu = (2^n)$. The configuration $\{\nu\}$ consists of diagrams $\nu^{(k)}, 1 \leq k \leq n-2$, and $|\nu^{(1)}| = 2n-4, |\nu^{(k)}| = 2n-2k-3, 2 \leq k \leq n-2$. From condition $P_m^{(k)}(\nu) \geq 0$ for all $m, k \geq 2$ it follows that the configuration $\{\nu\}$ may be considered as a path of length $n-3$ in some graph. The vertices of this graph correspond to the two-rows Young diagrams with $2k-1$ boxes and all ranges are

$$(k+1,k) \to (k,k-1),$$

$$(k+2+\alpha, k-1-\alpha) \to (k+\alpha, k-1-\alpha),$$

$$\alpha \geq 0,\ 1 \leq k \leq n-2.$$

Let us consider now the diagram $\nu^{(1)}$. It is easy to see that there are only two μ-admissible configurations $\tilde{\nu}$ and $\tilde{\tilde{\nu}}$:

$$\tilde{\nu} = \{\tilde{\nu}^{(1)} = (n-2, n-2), \tilde{\nu}^{(k)} = (n-1-k, n-2-k)\ ,\ 2 \leq k \leq n-2\}.$$

For the first configuration the non zero vacancies numbers are:

$$P_2^{(1)}(\tilde{\nu}) = 2n - 2(2n-4) + 2n - 7 = 1,\ P(2)_2(\tilde{\nu} = 1.$$

$$\tilde{\tilde{\nu}} = \{\tilde{\tilde{\nu}}^{(1)} = (n-2, n-3, 1), \tilde{\tilde{\nu}}^{(k)} = (n-1-k, n-2-k)\ ,\ 2 \leq k \leq n-2\}.$$

The charge $C(\tilde{\nu}) = 3$. In the last case the non zero vacancies numbers are: $P_1^{(1)}(\tilde{\tilde{\nu}}) = P_3^{(1)}(\tilde{\tilde{\nu}}) = 1, P_2^{(1)}(\tilde{\tilde{\nu}}) = 3$. The charge $C(\tilde{\tilde{\nu}}) = 5$. Consequently

$$k_{\lambda,\mu}(q) q^3 \begin{bmatrix} n-1 \\ 1 \end{bmatrix} \begin{bmatrix} n-3 \\ 1 \end{bmatrix} + q^5 \begin{bmatrix} 2 \\ 1 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} \begin{bmatrix} n-1 \\ 3 \end{bmatrix}. \tag{3.13}$$

The formula (3.13) is equivalent to the expression for $k_{\lambda,\mu}(q)$ from an example 5.3 [12]. Example under consideration illustrates the following general fact.

Let λ, μ be partitions. We denote by $\Omega(\lambda,\mu)$ the set of all μ-admissible λ-configurations.

THEOREM 3.3. *Let be $\lambda_n = prt_n(\alpha,\beta)$ (see [12]), $\mu_n = (\beta_1^n)$ and $\Omega_n = \Omega(\lambda_n, \mu_n)$. There exists a one-to-one correspondence $v_n : \Omega_n \to \Omega_{n+1}$ such that*

(i) $P_m^{(k)}(v_n(\{\nu\})) = P_m^{(k)}(\nu)$,

(ii) $C(v_n(\{\nu\})) = C(\{\nu\})$,

for all k, m and $\{\nu\} \in \Omega_n$. ∎

Example 8. The Clebsch-Gordan numbers.

We define the Clebsch-Gordan numbers $C^{\lambda}_{\mu\nu}$ by means of the formula

$$S_{\lambda\backslash\nu}(x) = \sum_{\mu} C^{\lambda}_{\mu\nu} S_{\mu}(x). \tag{3.15}$$

The Clebsch-Gordan number $C^{\lambda}_{\mu\nu}$ describes the multiplicity with which a given irreducible representation V_{λ} of Lie algebra $g\ell(n)$ appears in the tensor product $V_{\mu} \otimes V_{\nu}$. It may be found by using the Littlewood-Richardson rule [11], or the Gelfand-Zelevinsky rule [15], or the Kostant-Steinberg formula [24]. From Theorem 3.1 there follows a new rule for computation of the numbers $C^{\lambda}_{\mu\nu}$ in terms of configurations.

Let λ, μ be partitions, $l(\lambda) = r, l(\mu) = s$. Let us consider the diagrams $\Lambda = (\lambda_1 + \mu_1, \ldots, \lambda_r + \mu_1, \mu_1, \mu_2, \ldots, \mu_s)$, $\rho = (\mu_1^r)$ and the Kostka polynomials $K_{\Lambda\backslash\rho,\nu}(q)$. It is clear that

$$K_{\Lambda\backslash\rho,\nu}(0) = C^{\nu}_{\lambda,\mu} = C^{\Lambda}_{\rho,\nu}. \tag{3.16}$$

Consequently, the multiplicity $C^{\nu}_{\lambda,\mu}$ is equal to the number of ν-admissible $(\Lambda\backslash\rho)$-configurations of charge zero. A more detailed formulation of the result is given below.

THEOREM 3.4. The multiplicity $C^{\nu}_{\lambda,\mu}$ is equal to the number of arrays of integers $\{\beta^{(k)}_n \mid k \geq 0, n \geq 1\}$ which satisfy the following conditions

(i) every $\beta^{(k)}_n$ is equal to either zero, or one.

(ii) $$\sum_{k\geq 0} \beta^{(k)}_n = \nu'_n, \sum_{n\geq 1} \beta^{(k)}_n = \begin{cases} \lambda_{k+1}, & \text{if } 1 \leq k \leq r-1, \\ \mu_{k+1-r}, & \text{if } r \leq k \leq r+s. \end{cases}$$

(iii) for all $k \geq 1, m \geq 1$ the following inequalities hold

1) $$\sum_{j\geq k} (\beta^{(j)}_m - \beta^{(j)}_{m+1}) \geq 0,$$

2) $$\min(m, \mu_1)\delta_{k,r} + \sum_{j\leq m} (\beta^{(k-1)}_j - \beta^{(k)}_j) \geq 0.$$

4. PROOF OF THEOREM 3.1 IN THE CASE WHEN $\rho = 0, \eta = 0$.

4.1. Rigged configurations and quantum numbers

Let λ, ρ, μ, η be partitions such that $\rho \subseteq \lambda, |\lambda| - |\rho| = |\mu| + |\eta|, \rho = (m^l)$.

It will be convenient to change slightly the terminology introduced in Section 2.

A collection of Young diagrams $\{\nu^{(k)}\}$ such that

$$|\nu^{(k)}| = \sum_{j\geq k+1} = (\lambda_j - \rho_j - \eta'_j) \tag{4.1}$$

is called a $(\lambda\backslash\rho,\mu|\eta)$-configuration (or, in short, a configuration).

Let us define the vacancies numbers for a $(\lambda\backslash\rho,\mu|\eta)$-configuration $\{\nu\}$ by

$$\begin{aligned}P_n^{(k)}(\nu) &= P_n^{(k)}(\nu,\mu|\eta = \\ (4.2)\qquad &= \eta'_k - \eta'_{k+1} + \min(n,\rho_k - \rho_{k+1}) + \\ &+ Q_n(\nu^{(k-1)}) - 2Q_n(\nu(k)) + Q_n(\nu_{(k+1)}),\end{aligned}$$

with the initial condition $\nu^{(0)} = \mu$.

A configuration $\{\nu\}$ is called $(\mu|\eta)$-admissible if all the vacancies numbers $P_n^{(k)}(\nu,\mu|\eta)$ are nonnegative. Now we want to describe new combinatorial objects which in the sequel will be called the rigged configurations.

In order to rig a configuration we fill in the first collumns of the diagrams $\nu^{(k)}$ with quantum numbers $\mathcal{I}_{n,\alpha}^{(k)}, n \geq 1, 1 \leq k \leq l(\lambda), 1 \leq \alpha \leq m_n(\nu^{(k)})$ using the following rule: the extreme left boxes in the rows of length n of a diagram $\nu^{(k)}$ are filled in from top to bottom with a nondecreasing sequence of integers $\mathcal{I}_{n,\alpha}^{(k)}$ not exceeding $P_n^{(k)}(\nu)$:

$$(4.3)\qquad 0 \leq \mathcal{I}_{n,1}^{(k)} \leq \mathcal{I}_{n,2}^{(k)} \leq \ldots \leq \mathcal{I}_{n,m_n(\nu^{(k)})}^{(k)}.$$

Denote by $QM(\lambda\backslash\rho,\mu|\eta)$ the set of rigged $(\mu|\eta)$-admissible $(\lambda\backslash\rho,\mu|\eta)$-configurations. The elements of $QM(\lambda\backslash\rho,\mu|\eta)$ are called quantum arrays of type $(\lambda\backslash\rho,\mu|\eta)$ and are denoted by $(\{\nu\},\mathcal{I})$.

Example 4.1. Let λ,μ be partitions of the number $n, \rho = (0), \eta = (0)$. Put $n_k = \sum_{j\geq k+1}\lambda_j$. Assume that $\mu'_1 \geq n - \lambda_1 + \lambda_2$. Then λ-configuration $\{\nu\} = \{(1^{n_1}),(1^{n_2}),\ldots\}$ is μ-admissible. Indeed, all vacancies numbers are nonnegative, since $P_n^{(k)}(\nu) = 0$, if $n \geq 2, P_1^{(k)}(\nu) = \lambda_k - \lambda_{k+1}$, if $k \geq 2$ and by assumption $P_1^{(1)}(\nu) = \mu'_1 + \lambda_1 - \lambda_2 - n \geq 0$. The configuration charge $C(\{\nu\})$ is computed using (3.3) where we must set $\alpha_n^{(k)} = (\nu^{(k)})'_n$:

$$C(\{\nu\}) = C = n(\mu) + n(\lambda') - (n-\lambda_1)(n-\mu'_1) - \binom{n}{2}.$$

The contribution of the configuration $\{\nu\}$ to the Kostka polynomial is given by

$$q^c \begin{bmatrix} \mu'_1 - \lambda_2 \\ n_1 \end{bmatrix}_q \Pi_{k\geq 2} \left[n_k + \lambda_k - \lambda_{k+2}\right]_q.$$

Theorem 2.2 implies the following assertion.

THEOREM 4.1. *The number of bitableaux with shape $\lambda\backslash\rho$ and weight $\mu|\eta$ is equal to the number of type $(\lambda\backslash\rho,\mu|\eta)$ quantum arrays:*

$$(4.4)\qquad \text{Card}\,\left|SBY(\lambda\backslash\rho,\mu|\eta)\right| = \text{Card}\,\left|QM(\lambda\backslash\rho,\mu|\eta)\right|. \qquad \blacksquare$$

4.2. Combinatorial correspondence

Following the established terminology [6], the rows of a diagram $\nu^{(k)}$ belonging to a rigged configuration $(\{\nu\}, \mathcal{I})$ will be called strings of length n and type k. Thus with each string of length n and type k a quantum number $\mathcal{I}_{n,\alpha}^{(k)}$ not exceeding $P_n^{(k)}(\nu)$ is associated. A string is called special if the corresponding quantum number has the maximal possible value, i.e. $\mathcal{I}_{n,\alpha}^{(k)} = P_n^{(k)}(\nu)$.

In this present paper we shall construct only the bijections

$$QM(\lambda\backslash\rho, \mu) \underset{\pi_*}{\overset{\pi^*}{\rightleftharpoons}} STY(\lambda\backslash\rho, \mu). \tag{4.5}$$

The case of supertableaux is considered in [10].

IA. Costruction of the mapping π^*(case $\rho = 0$) . Observe that a Young diagram λ is uniquely reconstructed from a configuration $\{\nu\}$ and a weight μ. Thus we have to construct a tableau from the set $STY(\lambda, \mu)$ given a rigging $\mathcal{I}$ of $\{\nu\}$. To this end, we shall fill in the boxes of λ with numbers

$$\underbrace{p, \dots, p}_{\mu_p}, \underbrace{p-1, \dots, p-1}_{\mu_{p-1}}, \dots, \underbrace{1, \dots, 1}_{\mu_1} \tag{4.6}$$

where $\mu = (\mu_1, \dots, \mu_p)$.

It is sufficient to specify the following:

(i) Into which row of λ the number p should be placed.

(ii) Which admissible rigged configuration corresponds to the new diagram $\tilde{\lambda} = \lambda\backslash\{p\}$ and to the weight $\tilde{\mu} = \mu_1, \dots, \mu_{p-1}, \mu_p - 1)$.

With this goal in mind, let us consider the rigging $\mathcal{I}$ of $\{\nu\}$.

Let us define the rank of a rigging.

Put $\nu^{(0)} = \mu$. The rows of the diagram μ will be called type 0 strings. We assume that $\nu^{(0)}$ contains only one special string, which corresponds to the last row μ_p of $\mu, p = l(\mu)$. We say that rank $\mathcal{I}$ is equal to r if r is the greatest of all numbers r' such that for all $k, 0 \le k \le r$ there are special type k strings in $\{\nu\}$ whose lengths m_k satisfy the condition $m_0 = \mu_p \le m_1 \le \dots \le m_r$ and, moreover, the diagram $\nu^{(r+1)}$ does not contain special strings with length exceeding $m_r - 1$.

Let $QM^{(r)}(\lambda, \mu)$ be the set of all μ-admissible rank r rigged λ-configurations. Denote by $STY^{(r)}(\lambda, \mu)$ the set of all standard tableaux with shape λ and weight μ such that the number p makes its first appearence in the $(r+1)$-th row. Clearly,

$$QM(\lambda, \mu) = \coprod_{r \ge 0} QM^{(r)}(\lambda, \mu), STY(\lambda, \mu) = \coprod_{r \ge 0} STY^{(r)}(\lambda, \mu). \tag{4.7}$$

Let us define the mapping $\pi_r^* : QM^{(r)}(\lambda, \mu) \to STY^{(r)}(\lambda, \mu)$ as follows. Let $(\{\nu\}, \mathcal{I}) \in QM^{(r)}(\lambda, \mu)$. Then the number p is inscribed into the extreme right box of the row λ_{r+1}. The diagram $\widetilde{\lambda}$ is obtained from λ by removing this box. Put $\widetilde{\mu} = \mu_1, \dots, \mu_{p-1}, \mu_p - 1)$. Let us now pass to the description of the type $(\widetilde{\lambda}, \widetilde{\mu})$ quantum array $(\{\widetilde{\nu}\}, \widetilde{\mathcal{I}})$.

If $r = \operatorname{rank} \mathcal{I} = 0$ we put $(\{\widetilde{\nu}\}, \widetilde{\mathcal{I}}) = (\{\nu\}, \mathcal{I})$; in other words, if $r = 0$ all quantum numbers and the configuration remain the same. Assume that $r > 0$. Among the special strings with length greater or equal to μ_p of the diagram $\nu^{(1)}$ find one lying below all the others. Let $\Lambda^{(1)}_{m_1}$ be this string and m_1 its length. Assume that for some $k, 1 \le k < r$, the special strings $\Lambda^{(1)}_{m_1}, \dots, \Lambda^{(k)}_{m_k}$ are already constructed. Then $\Lambda^{(k+1)}_{m_{k+1}}$ is the special string lying below all the other special strings with length greater or equal to m_k in the diagram $\nu^{(k+1)}$. Thus we have constructed a sequence of special strings $\Lambda^{(k)}_{m_k} \in \nu^{(k)}, 1 \le k \le r$, such that there are no special strings lying between $\Lambda^{(k)}_{m_{k-1}}$ and $\Lambda^{(k)}_{m_k}$ as the rows of the diagrams $\nu^{(k)}$. Let $\widetilde{\nu}^{(k)}$ be the Young diagram obtained from $\nu^{(k)}$ by deleting the extreme right box from the row $\Lambda^{(k)}_{m_k} \in \nu^{(k)}, 1 \le k \le r$; if $k > r$ we set $\widetilde{\nu}^{(k)} = \nu^{(k)}$. We obtain a configuration $\{\widetilde{\nu}\}$ together with distinguished rows $\widetilde{\Lambda}^{(k)}_{m_k - 1} \in \widetilde{\nu}^{(k)}$ of length $m_k - 1$. Note that after passing from $\{\nu\}$ to $\{\widetilde{\nu}\}$ all strings $\Lambda^{(k)}_{m_k}$ with length $m_k = 1$ disappear. The rigging $\widetilde{\mathcal{I}}$ of $\{\widetilde{\nu}\}$ is defined as follows: the quantum numbers $\mathcal{I}^{(k)}_{n,\alpha}$ corresponding to all strings except $\widetilde{\Lambda}^{(k)}_{m_k-1}$ remain the same, i.e. $\widetilde{\mathcal{I}}^{(k)}_{n,\alpha} = \mathcal{I}^{(k)}_{n,\alpha}$ if $(n,\alpha) \ne m_k, m_{m_k}(\nu^{(k)})), 1 \le k \le r$; the strings $\widetilde{\Lambda}^{(k)}_{m_k-1}$ (for $m_k > 1$) are declared special and lie at the bottom of the block of sort k strings with length $m_k - 1$ (when $m_k = 1$ such strings disappear).

PROPOSITION 4.1. *The set $\widetilde{\mathcal{I}}^{(k)}_{n,\alpha}$ defines a rigging of the configuration $\{\widetilde{\nu}\}$.*

To prove this assertion let us trace down the changes of the vacancies numbers $P^{(k)}_n(\nu)$ with weight ν as we pass from (ν, μ) to $(\widetilde{\nu}, \widetilde{\mu})$. Put $m_{r+1} = m_{r+2} = \dots = \infty$.

Clearly, $Q_n(\widetilde{\nu}^{(k)}) = Q_n(\nu^{(k)}) - \theta(n - m_k)$ for all k, where

$$\theta(x) = \begin{cases} 1, & x \ge 0 \\ 0, & x < 0 \end{cases} \tag{4.8}$$

Consequently

$$P^{(k)}_n(\widetilde{\nu}) = P^{(k)}_n(\nu) + \begin{cases} 0, & \text{if } n < m_{k-1}, \\ -1, & \text{if } m_{k-1} \le n < m_k, \\ +1, & \text{if } m_k \le n < m_{k+1}, \\ 0, & \text{if } m_{k+1} \le n. \end{cases} \tag{4.9}$$

It remains to notice that by construction there are no special strings lying between $\Lambda^{(k)}_{m_{k-1}}$ and $\Lambda^{(k)}_{m_k}$, i.e. $\mathcal{I}^{(k)}_{n,\alpha} < P^{(k)}_n(\nu)$ for $m_{k-1} \le n < m_k$. So the proposition 4.1 is proved.

Notice that since $\mu_p \leq m_1 \leq \ldots \leq m_r$, if $\mu_p > 1$, we have

$$\text{rank}(\widetilde{\mathcal{I}}) \geq \text{rank}(\mathcal{I}). \tag{4.10}$$

Filling in recursively the diagram λ with numbers from the sequence (4.6) we end up with the λ-shaped tableau with weight μ. The inequality (4.10) implies that the resulting tableau is standard.

Thus we have defined the mapping π_r^* and hence also π^*.

I.B. Construction of the mapping π_* (case $\rho = 0$).

Let $T \in STY^{(r)}(\lambda, \mu)$. We shall construct the quantum array corresponding to the tableau T inductively by the number of boxes in λ. So let us suppose that π_* is already constructed for all tableaux with the number of boxes strictly less than $|\lambda|$. Denote by $\widetilde{T}$ the tableau obtained from T by deleting the extreme right box in the $(r+1)$-th row. Let $\widetilde{\lambda}, \widetilde{\mu}$ be its shape and weight. Denote by $(\widetilde{\nu}, \widetilde{\mathcal{I}})$ the type $\widetilde{\lambda}, \widetilde{\mu}$ quantum array which corresponds to $\widetilde{T}$ by the induction hypothesis. We wish to construct a rigged configuration $(\{\nu\}, \mathcal{I})$ in such a way that $\pi^*(\{\nu\}, \mathcal{I}) = T$. For this, let us reorder the quantum numbers $\widetilde{\mathcal{I}}_{n,\alpha}^{(k)}$ arranging them in the nonincreasing order from top to bottom within each block of sort k strings with length n. Thus the string with the maximal quantum number lies in the first row of such block. It is convenient to assume that diagram $\nu^{(k)}$ is formally enhanced with an extra $(l(\nu^{(k)}) + 1)$-th row of length zero which corresponds to a length zero special string $\Lambda_0^{(k)}$.

Construction of the configuration $\{\nu\}$.

If $r = 0$, we put $(\{\nu\}, \mathcal{I}) = (\{\widetilde{\nu}\}, \widetilde{\mathcal{I}})$. Formula (3.5) implies that rang $\mathcal{I} = 0$. Assume that $r > 0$. Consider a diagram $\widetilde{\nu}^{(r)} \in \{\widetilde{\nu}\}$. Find the first from the top special string $\widetilde{\Lambda}_{m_r}^{(r)} \in \widetilde{\nu}^{(r)}$ and let m_r be its length (possibly, $m_r = 0$). Assume that for some k, $1 < k \leq r$, we have already constructed special strings $\widetilde{\Lambda}_{m_r}^{(r)}, \ldots, \widetilde{\Lambda}_{m_k}^{(k)}$ such that $m_k \leq m_{k+1} \leq \ldots \leq m_r$. As the next string $\widetilde{\Lambda}_{m_{k-1}}^{(k-1)}$ let us take the first from the top type $(k-1)$ special string with length not exceeding m_k (possibly, $m_{k-1} = 0$). Thus we have constructed a sequence of special strings $\widetilde{\Lambda}_{m_k}^{(k)}$, $1 \leq k \leq r$, such that for all $1 \leq k \leq r$, there are no special strings lying between $\widetilde{\Lambda}_{m_{k+1}}^{(k)}$ and $\widetilde{\Lambda}_{m_k}^{(k)}$. From (3.6) it follows that $m_1 \geq \mu_p - 1$. Add one box on the right end of each string $\widetilde{\Lambda}_{m_k}^{(k)}$, $1 \leq k \leq r$. We obtain a new diagram $\nu^{(k)}$ with a distinguished row $\Lambda_{m_k+1}^{(k)}$ of length $m_k + 1$, $1 \leq k \leq r$. The set $\{\nu^{(k)}, 1 \leq k \leq r\} \cup \{\widetilde{\nu}^{(k)}, k > r\}$ defines a configuration $\{\nu\}$.

The rigging $\mathcal{I}$ of the configuration $\{\nu\}$ is defined as follows: the quantum numbers corresponding to all strings except $\widetilde{\Lambda}_{m_k}^{(k)}$ remain the same, i.e. $\mathcal{I}_{n,\alpha}^{(k)} = \widetilde{\mathcal{I}}_{n,\alpha}^{(k)}$ if $(n, \alpha) \neq (m_k, m_{m_k}(\widetilde{\nu}^{(k)}))$, $1 \leq k \leq r$. The strings $\Lambda_{m_k+1}^{(k)}$ are declared special. Then such strings are placed on the top of each block of type k strings with length $m_k + 1$.

PROPOSITION 4.2. *The set* $\mathcal{I}_{n,\alpha}^{(k)}$ *defines a rigging of* $\{\nu\}$. *Moreover*

$$\pi^*(\{\nu\},\mathcal{I}) = T$$ ■

Indeed, (4.8) implies that

$$P_n^{(k)}(\nu) = P_n^{(k)}(\widetilde{\nu}) + \begin{cases} 0, & \text{if } n < m_{k-1} \\ +1, & \text{if } m_{k-1} \le n < m_k \\ -1, & \text{if } m_k \le n < m_{k+1} \\ 0, & \text{if } m_{k+1} \le n \end{cases} \tag{4.11}$$

Finally, let us observe that by construction there are no special strings lying between $\widetilde{\Lambda}_{m_{k+1}}^{(k)}$ and $\widetilde{\Lambda}_{m_k}^{(k)}$, $1 \le k \le r$ i.e. $\widetilde{\mathcal{I}}_{n,\alpha}^{(k)} < P_n^{(k)}(\widetilde{\nu})$ for $m_k \le n < m_{k+1}, (n,\alpha) \neq (m_k, m_{m_k}(\widetilde{\nu}^{(k)}))$. Hence, the numbers $\mathcal{I}_{n,\alpha}^{(k)}$ define a rigging. Furthermore, (4.11) implies that the sequence of special strings $\Lambda_{m_k+1}^{(k)}$, $1 \le k \le r$, has the folowing properties:

(i) $m_0 \le m_1 + 1 \le \ldots \le m_r + 1$ and in the diagram $\nu^{(r+1)}$ there are no special strings with length greater then m_r. '

(ii) there are no special strings lying between $\Lambda_{m_{k-1}+1}^{(k)}$ and $\Lambda_{m_k+1}^{(k)}$, $1 \le k \le r$.

Hence, rang $\mathcal{I} = r$. By the induction hypothesis we may assume that $\pi^*(\{\widetilde{\nu}\},\widetilde{\mathcal{I}}) = \widetilde{T}$. Taking into account the properties (i), (ii) and the definition of π^* (see IA), we conclude that $\pi^*(\{\nu\},\mathcal{I}) = T$. Assertion 4.2 is proved.

Thus, a bijection $QM^{(r)}(\lambda,\mu) \rightleftharpoons STY^{(r)}(\lambda,\mu)$ is constructed.

By examining the correspondence (4.5) (for $\rho = 0$) one can show that it has the following properties which we describe in a series of lemmas.

LEMMA 4.1. *Let* $\rho = (m^l)$, $(\nu,\mathcal{I})$ *be a type* $(\lambda\backslash\rho,\mu)$ *quantum array. Then*

$$\pi^*(\{\nu\},\mathcal{I}) \in STY(\lambda\backslash\rho,\mu).$$ ■

LEMMA 4.2. *Let* μ *be a partition,* $l(\mu) = p, t \in STY(\lambda,\mu)$, *let* $\omega(T)$ *be the word corresponding to* T *and* $\omega_1, \omega_2, \ldots$ *be the standard subwords extracted from* $\omega(T)$ *by means of the Lascoux-Schützenberger algorithm for the calculation of the charge (see (1.3), Section 1). Let* $\{\nu\}$ *be the configuration which corresponds to* T *under mapping* π^*.

Then, if $k \le \mu_p$, *we have* $(\nu^{(1)})_k' = \mathrm{d}(\omega_k)$. ■

The definition $\alpha(\omega)$ is given in (1.4), Section 1.

LEMMA 4.3. *Let $T \in STY^{(r)}(\lambda,\mu)$, let $\widetilde{T}$ be the tableau obtained from T by deleting the extreme left box in the $(r+1)$-th row (see the definition of π_*). Let $(\{\widetilde{\nu}\},\widetilde{\mathcal{I}})$ be the quantum array which corresponds to $\widetilde{T}$. Then*

1. $$C(\{\nu\}) + \sum_{k,n,\alpha} \mathcal{I}_{n,\alpha}^{(k)} - C(\{\widetilde{\nu}\}) - \sum_{k,n,\alpha} \widetilde{\mathcal{I}}_{n,\alpha}^{(k)} = (\mu')_{\mu_p} - (\nu^{(1)})'_{\mu_p} - 1,$$

2. $$C(T) - C(\widetilde{T}) = (\mu')_{\mu_p} - \mathrm{d}(\omega - \mu_p) - 1.$$ ∎

COROLLARY 4.1. *The charges of T and $\{\nu\}$ and the set of quantum numbers $\mathcal{I}_{n,\alpha}^{(k)}$ associated with the tableau T are related by the formula*

$$C(T) = C(\{\nu\}) + \sum_{k,n,\alpha} \mathcal{I}_{n,\alpha}^{(k)}. \tag{4.12}$$ ∎

Note that (4.12) along with (1.14) imply Theorem 3.1 (for the case $\rho = 0, \eta = 0$).

Remark 4.1. Let $\mu = (\mu_1, \ldots, \mu_k)$ be a partition, let $\sigma \in S_k$ be a permutation and $\mu^\sigma = (\mu_{\sigma^{-1}(1)}, \ldots, \mu_{\sigma^{-1}(k)})$.

It is clear that $QM(\lambda,\mu^\sigma) = QM(\lambda,\mu)$. Hence, one can define the mappings φ_σ and $\varphi_{\sigma^{-1}}$ such that the following diagram is commutative

$$\begin{array}{ccc} QM(\lambda,\mu^\sigma) & = & QM(\lambda,\mu) \\ \uparrow\downarrow & & \uparrow\downarrow \\ STY(\lambda,\mu^\sigma) & \underset{\varphi_{\sigma^{-1}}}{\overset{\varphi_\sigma}{\rightleftarrows}} & STY(\lambda,\mu) \end{array} \tag{4.13}$$

It seems plausible that the map φ_σ concides with the Knuth transformation [22].

4.3. Involutions on the set of standard tableaux [10].

On the set $QM(\lambda\backslash\rho,\mu|\eta)$ we define the involution θ, which corresponds to «inversion of the quantum numbers»:

$$\theta(\{\nu\},\mathcal{I}) := (\{\nu\},\widetilde{\mathcal{I}}), \tag{4.14}$$

where $\widetilde{\mathcal{I}}_{n,\alpha}^{(k)} = P_n^{(k)}(\nu) - \mathcal{I}_{n,m_n(\nu^{(k)})-\alpha+1}^{(k)}$, $1 \le \alpha \le m_n(\nu^{(k)})$. Consequently, by virtue of (4.5) there exists an involution θ on the set $STY(\lambda,\mu)$. On the other hand, we defined in (1.6), Section 1, the Schützenberger involution

$$S : STY(\lambda,\mu) \to STY(\lambda,\overleftarrow{\mu}),$$

where $\overleftarrow{\mu} = (\mu_k, \mu_{k-1}, \ldots \mu_1)$ for $\mu = (\mu_1, \ldots \mu_k)$. Let us denote by φ the transformation $STY(\lambda,\overleftarrow{\mu} \to STY(\lambda,\mu)$ defined by diagram (4.13) with $\mu^\sigma = \overleftarrow{\mu}$.

THEOREM 4.2. *The following assertions hold*

1. *If* $\mu = (m_k)$ *, then* $S = \theta$ *.*
2. *The diagram below is commutative*

$$\begin{array}{ccc} STY(\lambda,\mu) & \xrightarrow{S} & STY(\lambda,\overleftarrow{\mu}) \\ & \searrow_{\theta} \qquad \swarrow_{\varphi} & \\ & STY(\lambda,\mu) & \end{array} \tag{4.15}$$ ∎

COROLLARY 4.2. *The Schützenberger involution* $S : STY(\lambda,(m^l))$ *transforms the charge functional into the index:*

$$C(S(T)) = \mathrm{Ind}(T). \tag{4.16}$$ ∎

The definition of the index for a tableaux with repetitions may be found in [10].

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